

Understanding Functions

Function As a Machine

“Some people think of a function as a machine that takes an input and produces an output”. Think about your CD Player as a machine that takes an input (for example an Elton John CD) and produces an output (i.e. ♪♪♪ La La ♪♪♪!!). Note that from such a machine you always expect the same output for a particular input. That is if you put a CD in your CD player and it plays a song, you expect to hear the same song whenever you put the same CD in the CD player. If you put your Elton John CD in the CD player, and it suddenly starts playing The Doors, you’ll probably have your CD player fixed!

When thinking of functions as machines, you can have the same concept in mind. They can get any kind of input and produce any sort of output, as long as that **for a particular input, they always produce the same output.**

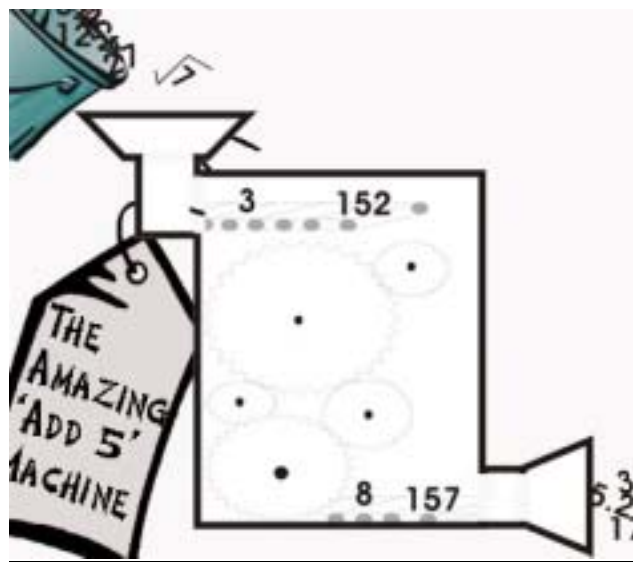


Figure 1- a machine that gets a number as input, adds 5 to it, and returns the result as output

Example

Imagine a function that gets a number as input, adds 5 to it, and returns the result as output (see figure 1):

- *If we give 3 as input, what will the output be?*

- *If we give 9 as input, what will the output be?*
- *If the output is 15, can you guess what the input has been?*

Domain and Range

For each machine, there is a variety of things that it can accept as input, and there is a variety of things that it can return as output. Your CD player, for instance, accepts all sort of audio CD's as input and for output it can produce a range of sounds for you to hear. You cannot play a videocassette with it because a videocassette is not in the set of things that your CD player accepts as input. Likewise, it sounds silly to expect your CD player to produce apple juice!

The above concept applies for functions as well. Each function has a set of things that it accepts as input. We call this set the **Domain** of a function. Also, for each function there is a set of things that it can return as output. We call this set the **Range** of a function.

Example

- *We can think of your CD player as a function that gets audio CDs as input and returns audio as output. Lets try to find the domain and range of it.*

Finding the domain of a function basically means to find a set of all of its possible inputs. Our CD player function can play any type of audio CD, so its domain will be the set of all of the audio CDs in the universe. Similarly, finding the range of a function means to find the set of all of its possible outputs. Our CD player function can only produce sounds, so the range of it would be the set of all audio sounds that it can produce.

- *Consider the function presented in the last page example. Try to find its domain and range.*

That function gets a number, adds 5 to it, and returns the answer. So lets make a set of its acceptable inputs:

$$\{\dots, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

At this point \mathbb{IN} (the set of all integers) seems like a reasonable domain for our function. But what if we try to give 0.45 as an input? The function's definition says that it "gets a number, as input..." It doesn't specify what kind of number the function gets, so 0.45 is an acceptable input. In the same fashion, this function can get any real number as its input, so its domain is \mathbb{IR} (the set of all real numbers). To find the range, remember that this function gets any real number, adds 5 to it, and returns it as output. Therefore its range is also the set of all real numbers (\mathbb{IR})¹.

¹ [Optional Reading] If you don't believe that this function can produce all real numbers, imagine \mathbf{a} is a real number that is not in the range of our function. Since \mathbf{a} is a real number, $\mathbf{b} = \mathbf{a} - 5$ is also a real number. Therefore \mathbf{b} is in the domain of the function and is an acceptable input. If we give \mathbf{b} as the input to the function, the function will return $\mathbf{b} + 5$. Remember that $\mathbf{b} = \mathbf{a} - 5$, so the output is equal to $(\mathbf{a} - 5) + 5 = \mathbf{a}$. This means that \mathbf{a} is in the set of the function's possible outputs, so \mathbf{a} is in the range of the function. This

The Function Notation (Part I)

Here is a simple notation by which we can represent a function in mathematics:

$$y = x^2$$

This equation mainly defines the relation between the input and output of the function, i.e. how this machine will manipulate the input (shown by 'x') to produce the output (indicated by 'y'). To understand the notation, you need to remember that a function is basically a machine that gets an input and gives back a corresponding output. In the above notation, we feed the input to the function through the variable x, and whatever y equals to will be the output of the function.

Example 1

If we want to give 7 as an input to the function and find the corresponding output, we have to let $x = 7$, and then find the value of y in the equation: $y = x^2$

Let's rewrite the function:

$$y = x^2$$

Now let $x = 7$, so we'll have:

$$y = (7)^2$$

$$y = 49 \quad \leftarrow \text{this is the output of the function, when the input is } 7$$

Example 2

What is the output of ' $y = 3x^2 + 4x - 11$ ', when $x = 13$?

Let's not let the ugly equation intimidate us! All we have to do is repeat the same steps as in Example 1:

- Rewrite the function:

$$y = 3x^2 + 4x - 11$$

- Now let $x = 13$, so we will have:

$$y = 3(13)^2 + 4(13) - 11$$

$$y = 548 \quad \leftarrow \text{this is the output of the function, when the input is } 13$$

shows that any real number that you can think of is in the range of the function, i.e. **IR** is the range of the function.

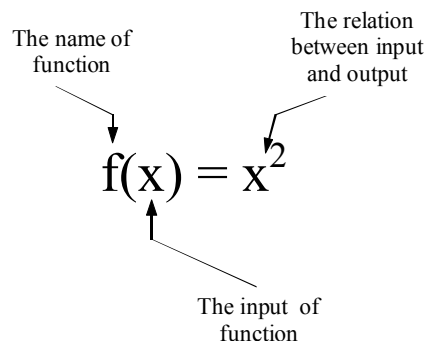
“y is a function of x.” What does that really mean?

You will come across this statement very often in mathematics. To understand it, once again you need to remember that a function is simply a machine that gets an input, and produces an output accordingly. Let's go back to our CD player function in which the audio CD's are the inputs and the sound you hear is the output. Note that the sound you will hear from your CD Player **ALWAYS** depends on the CD you put into it. That is, the song your CD player plays is not just a random pick, but absolutely depends on the CD you have put into it. You can describe this dependence relation by saying: The song that the CD Player plays *is a function of* the CD we put in it. Also note that the reverse is not true. That is, the CD in the CD player does not necessarily depend on the song that the CD player is playing. For instance, both of your “The Beatles 1” and “The 60's Greatest Hits” CD's may contain the “Yesterday” song, and when the player is playing the Yesterday song, either of them could be in the CD Player.

Let's examine a more algebraic example. Consider the function: $y = x^2$. This function gets a number as input through the ‘x’ variable, squares it, and returns the answer through the ‘y’ variable. Note that the value of ‘y’ **ALWAYS** depends on the value of what input you give to the function through ‘x’. When $x=2$, ‘y’ will be 4. When $x=3$, ‘y’ will be 9, and so on. We can describe this dependence by saying: **y is a function of x**.

The Function Notation (Part II)

Bellow is another notation that is vastly used to represent functions in mathematics:



- **The name of the function**
This is a letter that we assign to each function so we can identify it. The whole notation is read as “**f of x**”. Most function notations use the letter **f**, but note that

there is nothing magical about the letter f , and you can assign any letter to your functions. For example: $g(x)$, $u(x)$, $L(x)$.

- **The input of the function**

Just like the $y=x^2$ notation, we feed the input to the function through the variable x . For instance, If you want to find the output of the function when the input is 17.54, you have to let $x=17.54$, and evaluate what “ f of x ” will be equal to when x is 17.54.

- **The relation between input and output**

The right side of the equation determines the relation between the function’s input and the output that it will produce. For instance, $f(x) = \sin(x^2) + 3$ means that the function will get each input, square it, find the sin of its square, add 3 to the result, and return the result as its output.

Example 1

a) Find $A(2)$ if $A(r) = \pi r^2$ b) Try to describe the function $A(r) = \pi r^2$ as a machine: What seems to be its input? What is its output? c) Explain what $A(2)$ means.

a) *Finding $A(2)$ basically means to find the value of $A(r)$ when $r = 2$. To do that, we use the same approach as when we evaluated $y = x^2$ in the example above:*

First, we rewrite the function:

$$A(r) = \pi r^2$$

Now let $r = 2$, so we’ll replace all r ’s with 2:

$$A(2) = \pi(2)^2$$

$$A(2) = 4\pi \approx 12.6$$

b) *Remember that the area of a circle with the radius r is πr^2 . Thus, the function $A(r)$ can be interpreted as a fascinating machine that gets the radius of a circle, and returns its area.*

c) *$A(2)$ is the value of function A when $r = 2$. Based on the description of the function in part b, $A(2) \approx 12.6$ can be interpreted as the area of a circle with the radius of 2 units.*

Example 2

a) For the function $T(x) = x + 0.14x$, evaluate $T(290)$. b) Try to describe this function as a machine: What seems to be its input? What is its output? c) Explain what $T(290)$ means.

a) *Just as in the above example, to find $T(290)$ we need to evaluate $T(x)$ when $x=290$*

$$T(x) = x + 0.14x$$

Now let $x = 290$, so we'll replace all x 's with 290:

$$T(290) = 290 + 0.14(290)$$

$$T(290) = 330.6$$

b) *The function gets a value, adds 14% of it to itself, and returns the result. That extra 14% will probably ring a bell for most BC consumers: GST & PST! ☺ So this function could be interpreted as a machine that gets the price of a product before tax, and returns the price after tax.*

c) *According to how we interpreted the function in step b, $T(290)=330.6$ is how much a \$290 product really costs a customer, after tax.*

Exercises

1- True or False:

- a) --- If we know that $F(17) = 0$, then 17 is not in the domain of F.
- b) --- If $f(2) = 16$ and $g(2) = 16$, then f and g must be the same function
- c) --- If $L(9) = 24$, and $L(10) = 24$, then $L(x)$ is not a valid function.
- d) --- If the speed of a car can be determined by $V(t) = 5t + 3$, where variable t indicates the time, then we can say that the speed of that car is a function of time.

2- Evaluate each of the following functions with the input value given.

a) $y = -2x + 13$ when $x = -5$

b) $k(r) = r^3 - r^2 + r$ when $r = 2$

c) $F(x) = \frac{4x^2 - 1}{6 - x}$ when $x = -6$

3- A taxi company charges its clients based on the function: $P(t) = 0.5t + 2.5$ where t is how long the trip takes (in minutes) and $P(t)$ is in dollars. a) If a trip takes 30 minutes, how much will it cost the client? b) If a client has to pay \$7.5, how long has her trip taken?

4- [BONUS] What is the domain and range of the function $P(t)$, introduced in question three?

Answer Key (Short Answers)

- 1- a) False
b) False
c) False
d) True

- 2- a) $y = 23$
b) $k(2) = 6$
c) $F(x) = 143/12$

- 3- a) \$17.5
b) 10 minutes

- 4- Domain: all positive real numbers (\mathbb{R}^+)
Range: all positive real numbers (\mathbb{R}^+)

Answer Key (Detailed Answers)

- 1- a) $F(17) = 0$ means that if we give 17 to the function as input, it will give us 0 for output. This basically means that we CAN give 17 as input to the function. Note that 0 is just like any other real number, and it can be an acceptable output for a function. Therefore, 17 is in the set of acceptable inputs of the function, and the statement is False.
- b) If $f(2) = 16$ and $g(2) = 16$, that doesn't necessarily mean that f and g are the same function. For instance if $f(x) = 8x$ and $g(x) = x^4$, both $f(2)$ and $g(2)$ will be 16, but the functions are certainly not the same
- c) Remember that functions can get any kind of input and produce any sort of output, as long as that **for a particular input, they always produce the same output.** The function $L(x)$ is not breaking that promise, so the statement is false. Sure the function is giving back the same output for two different inputs, but that is not against the only rule that we made for functions. However, if for instance $L(10) = 24$ at some point, and $L(10) = 25$ at some other point, then $L(x)$ was not a valid function.
- d) At any point, the speed of that car (V) can be determined by $V(t) = 5t + 3$, and it always depends on t (time). Therefore we can say the speed of that car is a function of time, and the statement is True.
- 2- a) $y = -2x + 13$ when $x = -5$
 Replacing all x 's with -5
 $y = -2(-5) + 13$
 $y = 23$
- b) $k(r) = r^3 - r^2 + r$ when $r = 2$
 Replacing all r 's with 2
 $k(2) = (2)^3 - (2)^2 + (2)$
 $k(2) = 6$
- c) $F(x) = \frac{4x^2 - 1}{6 - x}$ when $x = -6$
 Replacing all x 's with -6
 $F(-6) = (4(-6)^2 - 1) / (6 - (-6))$
 $F(-6) = 143/12$

- 3- a) To find the answer we have to give 30 to the function as input and find its output, which basically means we have to evaluate $P(t)$ when $t = 30$.

$$P(t) = 0.5t + 2.5$$

$$P(30) = 0.5(30) + 2.5$$

$$P(30) = 17.5$$

So the trip will cost \$17.5

- b) Here we know the output of the function, and want to find what the input has been to result such output. Therefore, we basically want to find the value of t , when $P(t) = 7.5$

$$\text{We know that: } P(t) = 0.5t + 2.5 \text{ and } P(t) = 7.5$$

$$P(t) = 0.5t + 2.5 = 7.5$$

$$0.5t + 2.5 = 7.5$$

$$0.5t = 5$$

$$t = 10$$

- 4- This is a little bit tricky. At first the function $P(t) = 0.5t + 2.5$ seems to accept all real numbers and also produce all real numbers. So one's first guess would be set of all real numbers (\mathbb{R}) for both its domain and range. However, remember that the input of the function is actually how long a trip takes in minutes. Hence, getting a negative number as the input makes no sense. The domain of this function, therefore, will be all positive real numbers (\mathbb{R}^+). In the same fashion, charging a negative amount of money is senseless, and the range of P will also be all positive real numbers (\mathbb{R}^+)².

² [OPTIONAL READING] One may argue that since the output of the function is in dollars, it cannot have "any" real value, and it has to be a multiple of cents. For instance paying \$3.1415 is impossible in real life. While this is a valid argument, this handout has left out this kind of details for simplicity.